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A Huygen's principle for anisotropic media

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Abstract. General results are developed for the Green's function appropriate to radiation or diffraction in anisotropic media. The formulae may be evaluated from a knowledge of the geometry of the wavevector surfaces. The results allow the calculation of radiation fields in anisotropic media in both the Fresnel and Fraunhofer region. The case of optical radiation in a uniaxial medium is treated in detail.

1. Introduction

The theory of radiation or diffraction of electromagnetic or acoustic waves is now well established for media which are isotropic (Born and Wolf 1964). The radiation field produced by a given source distribution may be readily expressed in terms of an integral over the source of the Green's function, or radiation field produced by a unit point source, multiplied by the weighting function appropriate to the source. Diffraction by an aperture large compared with the wavelength of the radiation may be reduced to the problem of calculating the radiation from an effective source distribution across the aperture, this reduction being known as Huygen's principle.

For an extended source or aperture of halfwidth a radiating at wavelength λ into an isotropic environment it is usual to divide the field into several regions depending on the distance r of the observation point P from the source or aperture. We shall always assume that $a \gg \lambda$ since it is not possible to formulate Huygen's principle when this is not satisfied (Sommerfeld 1954). It should be noted, however, that this condition need not hold for the theory of radiation from a source, although in most practical cases it is satisfied. If the distance r from P to the nearest point on the source is such that $r \leqslant \lambda$ then P is in the near field of the radiator. From $r > \lambda$, P is in the radiation zone. The radiation may be further analysed as follows. For $a \leqslant r \leqslant a^2/\lambda$, P is said to be in the Fresnel region, for $r > a^2/\lambda$, P is in the far field or Fraunhofer region. The region $\lambda < r < a$ does not seem to have any particular name.

In recent years the study of the propagation of both electromagnetic and acoustic waves in anisotropic crystals has become topical. This is currently most noticeable in the present widespread interest in nonlinear optics (Butcher 1965) and acoustic surface waves and their technological applications (White 1967). In order to understand the detailed behaviour of radiation from a source distribution or, equivalently, diffraction by an aperture, in an anisotropic environment, it is fundamental to derive the appropriate Green's function or, what is in essence almost equivalent, the effective Huygen's principle.

In a basic paper on the theory of wave motion in an anisotropic environment Lighthill (1960) laid the foundations for the calculation of the radiation from a given source distribution in an anisotropic medium. This work was later used by Buchwald (1961) to obtain results for acoustic surface wave radiation on anisotropic surfaces. Unfortunately, these authors stopped short of deriving the Green's function or Huygen's principle for the radiation and obtained results valid only for the

Fraunhofer region of the radiation field. In many cases, particularly in the current study of acoustic surface wave propagation, most interest is in the Fresnel region of the radiation field. It is therefore highly desirable to have results which are sufficiently general to be applicable in this régime. Recently, Bergstein and Zachos (1966) derived the effective Huygen's principle for electromagnetic radiation in a uniaxially anisotropic medium. This result is valid for $r > \lambda$. It is the purpose of the present work to use the techniques developed by Lighthill (1960) in conjunction with standard Green's function theory to derive a generalization of this result applicable to radiation and diffraction of electromagnetic or acoustic waves in materials with arbitrary anisotropy valid in the field region $r > \lambda$. Although this is a straightforward extension of Lighthill's work it does not appear to have been done in the literature to the best of the author's knowledge.

In §§ 2 and 3 we derive general expressions for the Green's functions appropriate to radiation and diffraction in an anisotropic medium in terms of multiple Fourier integrals. In § 4 we simplify the expressions involved using the principle of stationary phase to find formulae valid in the field region $r > \lambda$. In § 5 the general results are applied to the case of uniaxial anisotropy considered by Bergstein and Zachos.

2. Radiation from a source distribution

In this section we shall briefly consider the radiation from a source distribution $S(\boldsymbol{r})$ assumed to be oscillating sinusoidally in time with fixed frequency ω in a system governed by the general vector wave equation

$$\mathbf{D}\left(\frac{\partial}{\partial \boldsymbol{r}}\right) . A(\boldsymbol{r}) = S(\boldsymbol{r}) \tag{1}$$

where $\mathbf{D}(\partial/\partial \boldsymbol{r})$ is a linear tensor differential operator and $A(\boldsymbol{r})$ is the vector field amplitude, together with the radiation boundary condition that $A(\boldsymbol{r})$ for $r \to \infty$ has the form of an outgoing travelling wave which falls off radially at least as $1/r$. In the case of elastic wave propagation A would represent particle displacement, while in the electromagnetic case A might be the electric field vector.

To solve (1) it is usual (Morse and Feshbach 1953) to consider the related equation, with the same boundary condition

$$\mathbf{D} . \mathbf{G}(\boldsymbol{r}-\boldsymbol{r}') = \mathbf{I}\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{2}$$

where $\mathbf{G}(\boldsymbol{r}-\boldsymbol{r}')$, the tensor Green's function, is a function of $\boldsymbol{r}-\boldsymbol{r}'$ because of the translational invariance of $\mathbf{D}$, and $\mathbf{I}$ is the unit tensor. The solution of (1) is then given by

$$A(\boldsymbol{r}) = \int \mathbf{G}(\boldsymbol{r}-\boldsymbol{r}') . S(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}' \tag{3}$$

where the integral extends over the source region. If we introduce the Fourier transforms of the various quantities using the prescription

$$\mathbf{G}(\boldsymbol{k}) = \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k} \exp\{-\mathrm{i}\boldsymbol{k} . (\boldsymbol{r}-\boldsymbol{r}')\}\, \mathbf{G}(\boldsymbol{r}-\boldsymbol{r}') \tag{4}$$

we find from (2), on Fourier transforming and inverting, that

$$\mathbf{G}(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\boldsymbol{k}\, \mathbf{D}^{\mathrm{adj}}(\mathrm{i}\boldsymbol{k})}{|\mathbf{D}(\mathrm{i}\boldsymbol{k})|} \exp\{\mathrm{i}\boldsymbol{k} . (\boldsymbol{r}-\boldsymbol{r}')\} \tag{5}$$

where $\mathbf{D}^{\mathrm{adj}}$ is the adjoint matrix to $\mathbf{D}$ and $|\mathbf{D}|$ is the determinant of $\mathbf{D}$. In general $|\mathbf{D}(\mathrm{i}\boldsymbol{k})|$ will have several zeros corresponding to the various branches of the dispersion relation $|\mathbf{D}(\mathrm{i}\boldsymbol{k}, \omega)| = 0$. These zeros give rise to poles in the integrand of (5). The integral with respect to k_z in (5) may be performed using contour integration. Care must be taken in choosing the path of integration to satisfy the radiation condition, that is, to include only waves travelling out from the source point. Equation (5) becomes

$$\mathbf{G}(\boldsymbol{r}-\boldsymbol{r}') = \frac{\mathrm{i}}{(2\pi)^2}\int\int \mathrm{d}k_x\,\mathrm{d}k_y \sum_j \frac{\mathbf{D}^{\mathrm{adj}}\{\mathrm{i}k_x, \mathrm{i}k_y, \mathrm{i}k_z^j(k_x, k_y)\}}{\{\partial|\mathbf{D}(\mathrm{i}\boldsymbol{k})|/\partial k_z\}_{k_z^j(k_x,k_y)}}$$
$$\times \exp[\mathrm{i}\{k_x(x-x')+k_y(y-y')+k_z^j(k_x, k_y)(z-z')\}] \qquad (6)$$

where the summation is over the various branches of the dispersion relation $k_z = k_z(k_x, k_y, \omega)$ and the integration is taken over the wavevectors satisfying the radiation condition that $\boldsymbol{k}\,.\,\boldsymbol{r} > 0$. We defer further simplification of this until § 4.

In certain degenerate cases $\mathbf{D}^{\mathrm{adj}}(\mathrm{i}\boldsymbol{k})$ will have a zero corresponding to a zero of $|\mathbf{D}(\mathrm{i}\boldsymbol{k})|$. This situation occurs for elastic wave propagation in an isotropic material where the transverse modes are degenerate. In this case the sum over branches will go from 1 to 2 and the zero must be removed from $\mathbf{D}^{\mathrm{adj}}$ and cancelled.

3. Diffraction by a plane aperture

The problem of the diffraction of an incident wave by a plane aperture may be reduced to the evaluation of an integral similar to (6) for $a \gg \lambda$. We must solve equation (1) with zero on the right hand side subject to the boundary conditions that the solution takes the value $A_{\mathrm{inc}}(x, y)$ (where $A_{\mathrm{inc}}(x, y)$ is the disturbance produced by the incident wave on the aperture, which we take to lie in the plane $z = 0$) on the aperture, and is zero for the rest of the plane containing the aperture. Taking Fourier transforms we immediately find that $\mathbf{D}(\mathrm{i}\boldsymbol{k})\,.\,A(\boldsymbol{k}) = 0$, and it therefore follows that

$$|\mathbf{D}(\mathrm{i}\boldsymbol{k}, \omega)| = 0 \quad \text{and} \quad A(\boldsymbol{k}) = \sum_j W^j A^j(\boldsymbol{k})\,\delta\{k_z - k_z^j(k_x, k_y)\}$$

where $A^j(\boldsymbol{k})$ is the normalized eigenvector corresponding to the jth branch of the dispersion relation $k_z = k_z(k_x, k_y, \omega)$ and W^j is a weighting factor. It follows immediately on inverting the Fourier transform that

$$A(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int \mathrm{d}k_x\,\mathrm{d}k_y \sum_j W^j A^j(\boldsymbol{k}) \exp[\mathrm{i}\{k_x x + k_y y + k_z^j(k_x, k_y) z\}]. \qquad (7)$$

On the aperture plane $z = 0$, from (7) we have

$$A_{\mathrm{inc}}(x, y) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int \mathrm{d}k_x\,\mathrm{d}k_y \sum_j W^j A^j(\boldsymbol{k}) \exp\{\mathrm{i}(k_x x + k_y y)\}. \qquad (8)$$

On inverting the Fourier transform (8) and using the result to evaluate W^j, we find that

$$A(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{r}'\,\mathbf{G}^{\mathrm{Ap}}(\boldsymbol{r}-\boldsymbol{r}')\,.\,A_{\mathrm{inc}}(\boldsymbol{r}') \qquad (9)$$

where

$$\mathbf{G}^{\mathrm{Ap}}(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{(2\pi)^2}\int\int \mathrm{d}k_x\,\mathrm{d}k_y \sum_j \frac{\boldsymbol{A}^j(\boldsymbol{k})\boldsymbol{B}^j(\boldsymbol{k})}{\boldsymbol{A}^j(\boldsymbol{k})\,.\,\boldsymbol{B}^j(\boldsymbol{k})} \times \exp[\mathrm{i}\{k_x(x-x')+k_y(y-y')+k_z^j(k_x,k_y)z\}] \tag{10}$$

and $\boldsymbol{B}^j$ is the unit vector orthogonal to $\boldsymbol{A}^m$ for $m \neq j$.

Comparison with (3) shows that the problem of diffraction by an aperture may be formally regarded as equivalent to radiation from a source $\boldsymbol{A}_{\mathrm{inc}}(x, y)$, with an aperture tensor Green's function given by (10). Equation (9) is the generalization of Huygen's Principle. However, to be of any use the Green's functions given by (10) and (6) must be simplified. We do this in the next section.

4. Simplification of the Green's function

In most cases of interest we require a knowledge of the field in the radiation zone $|\boldsymbol{r}-\boldsymbol{r}'| \gg \lambda$. In this region it is possible to simplify (6) and (10) considerably using the principle of stationary phase (Born and Wolf 1964). According to this, for $|\boldsymbol{r}-\boldsymbol{r}'| \gg \lambda$, the dominant contributions to the integrals in (6) and (10) come from the vicinity of points on each branch where the phase

$$\mathrm{i}\boldsymbol{k}^j\,.\,(\boldsymbol{r}-\boldsymbol{r}') = \mathrm{i}|\boldsymbol{r}-\boldsymbol{r}'|\phi^j(k_x, k_y, \boldsymbol{r}-\boldsymbol{r}')$$

is stationary with respect to variations in $\boldsymbol{k}$ on the wavevector surfaces given by $|\mathbf{D}(\mathrm{i}\boldsymbol{k}, \omega)| = 0$, that is, at points where $\partial\phi^j/\partial k_x = \partial\phi^j/\partial k_y = 0$. In the region of such a point $\boldsymbol{k}^{\mathrm{s}}$ the phase may be expanded as follows (for convenience we omit the dependence on $\boldsymbol{r}-\boldsymbol{r}'$ since this is held fixed):

$$\phi^j(k_x, k_y) = \phi^j(k_x^{\mathrm{s}}, k_y^{\mathrm{s}}) + \tfrac{1}{2}\{\alpha_j(k_x-k_x^{\mathrm{s}})^2 + \beta_j(k_y-k_y^{\mathrm{s}})^2 + 2\gamma_j(k_x-k_x^{\mathrm{s}})(k_y-k_y^{\mathrm{s}})\} + \ldots \tag{11}$$

where

$$\alpha_j = \left(\frac{\partial^2\phi^j}{\partial k_x^2}\right)_s \qquad \beta_j = \left(\frac{\partial^2\phi^j}{\partial k_y^2}\right)_s \qquad \gamma_j = \left(\frac{\partial^2\phi^j}{\partial k_x\partial k_y}\right)_s.$$

Using the result that

$$\int_{-\infty}^{\infty}\int \mathrm{d}k_x\,\mathrm{d}k_y \exp\{\tfrac{1}{2}\mathrm{i}|\boldsymbol{r}-\boldsymbol{r}'|(\alpha_j k_x^2 + \beta_j k_y^2 + 2\gamma_j k_x k_y)\} = \frac{2\pi\mathrm{i}\sigma_j}{(|\alpha_j\beta_j-\gamma_j^2|)^{1/2}|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{2\pi\mathrm{i}\sigma_j}{|K_j|^{1/2}|\boldsymbol{r}-\boldsymbol{r}'|} \tag{12}$$

where

$$\begin{aligned}\sigma_j &= 1 \text{ for } \alpha_j\beta_j > \gamma_j^2,\ \alpha_j > 0\\ &\quad -1 \text{ for } \alpha_j\beta_j > \gamma_j^2,\ \alpha_j < 0\\ &\quad -\mathrm{i} \text{ for } \alpha_j\beta_j < \gamma_j^2\end{aligned}$$

and $K_j = \alpha_j\beta_j - \gamma_j^2$ is the Gaussian curvature of the surface $\phi^j = \phi^j(k_x, k_y)$, in (6) and (10) we find that

$$\mathbf{G}(\boldsymbol{r}-\boldsymbol{r}') = -\frac{1}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|}\sum_{j,s}\frac{\mathbf{D}^{\mathrm{adj}}(\mathrm{i}\boldsymbol{k}_j^{\mathrm{s}})\sigma_j^{\mathrm{s}}\exp\{\mathrm{i}\boldsymbol{k}_j^{\mathrm{s}}\,.\,(\boldsymbol{r}-\boldsymbol{r}')\}}{\{\partial|\mathbf{D}(\mathrm{i}\boldsymbol{k})|/\partial k_z\}_j^{\mathrm{s}}\,|K_j^{\mathrm{s}}|^{1/2}} \tag{13}$$

and

$$\mathbf{G}^{\mathrm{Ap}}(\boldsymbol{r}-\boldsymbol{r}') = \frac{\mathrm{i}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|}\sum_{j,s}\frac{\boldsymbol{A}_j^s\boldsymbol{B}_j^s\sigma_j^s\exp\{\mathrm{i}\boldsymbol{k}_j^s\,.\,(\boldsymbol{r}-\boldsymbol{r}')\}}{(\boldsymbol{A}_j^s\,.\,\boldsymbol{B}_j^s)|K_j^s|^{1/2}} \tag{14}$$

where the index j runs over the various branches of the dispersion relation and the index $s = s(j)$ labels the points of stationary phase on the jth branch. Equations (13) and (14), together with (3) and (9), provide the required generalizations of the radiation field from a source and Huygen's principle in anisotropic media for $|\boldsymbol{r}-\boldsymbol{r}'| \gg \lambda_j^s$, where λ_j^s is the wavelength corresponding to the smallest $\boldsymbol{k}_j^s$ in the sums (13) and (14). We note here that further approximations in the evaluation of the integrals (3) and (9) lead to the classification of the radiation field into Fresnel and Fraunhofer regions.

Equations (13) and (14) may be more conveniently expressed in terms of the geometrical properties of the wavevector surfaces $|\mathbf{D}(\mathrm{i}\boldsymbol{k})| = 0$ or equivalently $k_z = k_z^j(k_x, k_y)$. Since

$$\phi^j(k_x, k_y) = \frac{k_x(x-x')+k_y(y-y')+k_z^j(k_x, k_y)(z-z')}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

it immediately follows that, at a stationary phase point,

$$\frac{\partial k_z^j}{\partial k_x} = -\frac{x-x'}{z-z'}$$

$$\frac{\partial k_z^j}{\partial k_y} = -\frac{y-y'}{z-z'}$$

that is, the normal to the wavevector surface is parallel to $\boldsymbol{r}-\boldsymbol{r}'$, hence the Gaussian curvature of the wavevector surface $k_z = k_z^j(k_x, k_y)$ is given by (Salmon 1912)

$$\kappa^j = \left\{\frac{\partial^2 k_z^j}{\partial k_x^2}\frac{\partial^2 k_z^j}{\partial k_y^2} - \left(\frac{\partial^2 k_z^j}{\partial k_x \partial k_y}\right)^2\right\}\Big/\left\{1+\left(\frac{\partial k_z^j}{\partial k_x}\right)^2+\left(\frac{\partial k_z^j}{\partial k_y}\right)^2\right\}^2 = K_j\frac{(z-z')^2}{|\boldsymbol{r}-\boldsymbol{r}'|^2}. \tag{15}$$

Similarly at a stationary phase point

$$\frac{\partial|\mathbf{D}(\mathrm{i}\boldsymbol{k})|}{\partial k_z} = \frac{|\nabla|\mathbf{D}(\mathrm{i}\boldsymbol{k})||}{\{1+(\partial k_z/\partial k_x)^2+(\partial k_z/\partial k_y)^2\}^{1/2}} = |\nabla|\mathbf{D}(\mathrm{i}\boldsymbol{k})||\frac{(z-z')}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{16}$$

Hence we find, using (15) and (16) in (13) and (14),

$$\mathbf{G}(\boldsymbol{r}-\boldsymbol{r}') = -\frac{1}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|}\sum_{j,s}\frac{\mathbf{D}^{\mathrm{adj}}(\mathrm{i}\boldsymbol{k}_j^s)\sigma_j^s\exp\{\mathrm{i}\boldsymbol{k}_j^s\,.\,(\boldsymbol{r}-\boldsymbol{r}')\}}{|\nabla|\mathbf{D}(\mathrm{i}\boldsymbol{k})||_j^s|\kappa_j^s|^{1/2}} \tag{17}$$

and

$$\mathbf{G}^{\mathrm{Ap}}(\boldsymbol{r}-\boldsymbol{r}') = \frac{\mathrm{i}\boldsymbol{n}\,.\,(\boldsymbol{r}-\boldsymbol{r}')}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|^2}\sum_{j,s}\frac{\boldsymbol{A}_j^s\boldsymbol{B}_j^s\sigma_j^s\exp\{\mathrm{i}\boldsymbol{k}_j^s\,.\,(\boldsymbol{r}-\boldsymbol{r}')\}}{(\boldsymbol{A}_j^s\,.\,\boldsymbol{B}_j^s)|\kappa_j^s|^{1/2}} \tag{18}$$

where $\boldsymbol{n}$ is the unit normal to the aperture plane.

$\mathbf{G}$ and $\mathbf{G}^{\mathrm{Ap}}$ may be readily calculated from a knowledge of the operator $\mathbf{D}(\mathrm{i}\boldsymbol{k})$ and in conjunction with equations (3) and (9) allow a calculation of the radiation field for all $|\boldsymbol{r}-\boldsymbol{r}'| \gg \lambda$ to be reduced to an integration over the source region. In many cases this may be performed numerically. If the Gaussian curvature of the wavevector surface vanishes then we must retain higher order terms in the phase expansion (11). In general the effect of zero curvature is to make the Green's functions (17) and (18) fall off less rapidly with $|\boldsymbol{r}-\boldsymbol{r}'|$. The details of the radial dependence, however, depend on the shape of the surface in the neighbourhood of the stationary phase point. We do not consider this problem here since to do so in three dimensions with generality is extremely complex.

Results similar to equations (17) and (18) may easily be obtained in two dimensions. The particular case of acoustic surface waves is slightly more complicated owing to the need for additional boundary conditions to be satisfied. This case will be dealt with separately elsewhere.

5. Diffraction in a uniaxial material

In this section we apply the general results (17) and (18) to the particular case of optical diffraction in a uniaxial material. This case has already been studied by Bergstein and Zachos (1966), and we merely use this example as an illustration of the use of the general formulae.

In the forms (17) and (18) the tensor Green's functions do not depend on the choice of coordinate system. We therefore choose our coordinate system initially to coincide with the principal axes of the medium, with the z axis along the optic axis. We shall later rotate the results to transfer to a coordinate system where the diffracting aperture lies on $z = 0$.

After Fourier transformation Maxwell's equations yield

$$\boldsymbol{k}(\boldsymbol{k}\,.\,\mathscr{E})-k^2\mathscr{E}+\frac{\omega^2}{V^2}\boldsymbol{\epsilon}_r\,.\,\mathscr{E}=0 \tag{19}$$

where $\mathscr{E}$ is the electric field vector and

$$\boldsymbol{\epsilon}_r=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \epsilon_r\end{pmatrix}$$

where ϵ_r is the relative dielectric constant for propagation normal to the optic axis and V is the propagation velocity of the ordinary wave. Equation (19) immediately yields the dispersion relation

$$\left(k^2-\frac{\omega^2}{V^2}\right)\left(\frac{k_x{}^2+k_y{}^2}{\epsilon_r}+k_z{}^2-\frac{\omega^2}{V^2}\right)=0. \tag{20}$$

The eigenvectors corresponding to the two branches of the dispersion relation (20) are found to be, after some algebra,

$$\text{Ordinary} \qquad\qquad \text{Extraordinary}$$

$$\mathbf{O}=\begin{pmatrix}\dfrac{k_y}{P_1}\\[2ex] \dfrac{k_x}{P_1}\\[2ex] 0\end{pmatrix} \qquad\qquad \boldsymbol{E}=\begin{pmatrix}-\dfrac{\epsilon_r k_x k_z}{P_1P_2}\\[2ex] -\dfrac{\epsilon_r k_y k_z}{P_1P_2}\\[2ex] \dfrac{P_1}{P_2}\end{pmatrix} \tag{21}$$

where $P_1 = (k_x^2+k_y^2)^{1/2}$ and $P_2 = (k_x^2+k_y^2+\epsilon_r k_z^2)^{1/2}$. It is clear that $\mathbf{0} \,.\, \boldsymbol{E} = 0$ and $\mathbf{0} \,.\, \boldsymbol{k} = 0$. In order to apply (18) we must first find the stationary phase points where $\partial\phi^j/\partial k_x = \partial\phi^j/\partial k_y = 0$. Using (19) to evaluate the phase we find that

$$\boldsymbol{k}^{s0} = \frac{\omega}{V}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

and

$$k^{se}_{(x,y)} = \frac{\omega}{V}\epsilon_r^{1/2}\frac{(x-x', y-y')}{|\boldsymbol{r}-\boldsymbol{r}'|_e} \qquad k^{se}_z = \frac{\omega}{V\epsilon^{1/2}}\frac{z-z'}{|\boldsymbol{r}-\boldsymbol{r}'|_e}$$

where o and e signify ordinary and extraordinary, and

$$|\boldsymbol{r}-\boldsymbol{r}'|_e^2 = |\boldsymbol{r}-\boldsymbol{r}'|^2 - \frac{(\epsilon_r-1)}{\epsilon_r}(z-z')^2$$

is an effective distance between source and observation point. From (15) the square roots of the curvatures of the wavevector surfaces at the points $\boldsymbol{k}^s$ are easily found to be $|\kappa_s{}^0|^{1/2} = V/\omega$ and $|\kappa_s{}^e|^{1/2} = (V/\omega)|\boldsymbol{r}-\boldsymbol{r}'|_e^2/|\boldsymbol{r}-\boldsymbol{r}'|^2$. The eigenvectors $\boldsymbol{A}^j$ may be identified with $\mathbf{0}$ and $\boldsymbol{E}$ and a complete set for expansion may be made up by taking the unit vector perpendicular to $\mathbf{0}$ and $\boldsymbol{E}$. In this case the orthogonal vector set $\boldsymbol{B}^j$ will be identical with $\boldsymbol{A}^j$.

Using these results in (18) we find for the aperture tensor Green's function

$$\mathbf{G}^{\mathrm{Ap}}(\boldsymbol{r}-\boldsymbol{r}') = \frac{k_0\boldsymbol{n}\,.\,(\boldsymbol{r}-\boldsymbol{r}')\,\mathbf{0}(\boldsymbol{k}^{s0})\,\mathbf{0}(\boldsymbol{k}^{s0})\exp(ik_0|\boldsymbol{r}-\boldsymbol{r}'|)}{2\pi i|\boldsymbol{r}-\boldsymbol{r}'|\,|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{k_0\boldsymbol{n}\,.\,(\boldsymbol{r}-\boldsymbol{r}')\,\boldsymbol{E}(\boldsymbol{k}^{se})\,\boldsymbol{E}(\boldsymbol{k}^{se})\exp(ik_e|\boldsymbol{r}-\boldsymbol{r}'|_e)}{2\pi i|\boldsymbol{r}-\boldsymbol{r}'|_e\,|\boldsymbol{r}-\boldsymbol{r}'|_e} \tag{22}$$

where $k_0 = \omega/V$, $k_e = (\omega/V)\epsilon_r^{1/2}$, and $\mathbf{0}(\boldsymbol{k}^{s0})$, $\boldsymbol{E}(\boldsymbol{k}^{se})$ may be easily evaluated using the results above. The radiation Green's function may easily be found in a similar fashion but we do not quote the result here since it is similar to (22). Taking

$$|\boldsymbol{r}-\boldsymbol{r}'|_e^2 = |\boldsymbol{r}-\boldsymbol{r}'|^2 - \left(\frac{\epsilon_r-1}{\epsilon_r}\right)\{\hat{\boldsymbol{z}}\,.\,(\boldsymbol{r}-\boldsymbol{r}')\}^2$$

where $\hat{\boldsymbol{z}}$ is the unit vector along the optic axis we see that the form (22) is true for any coordinate system provided the polarization vectors $\boldsymbol{E}$ and $\mathbf{0}$ are appropriately rotated. The second term in (22) for the extraordinary wave is identical with Bergstein and Zachos' result except for the inclusion of polarization factors.

Similar calculations may be done for other symmetries. These will be presented elsewhere.

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